Multiple choice questions

Q1 (B)

Q2 (A), Equations of directrices $x = \pm \frac{a}{e} = \pm \frac{a^2}{\sqrt{a^2+b^2}}$

$= \pm \frac{25}{\sqrt{25+4}} = \pm \frac{25}{\sqrt{29}}$

Q3 (D), since $|z|<1, |z^2|<|z|$, and $\arg(z^2) = 2\arg(z)$

Q4 (A), Let $x = \alpha + 1, \therefore \alpha = x-1, \therefore 4(x-1)^3 + (x-1)^2$

$-3(x-1) + 5 = 4x^3 - 11x^2 + \ldots$

Q5 (B), z lies between 2 circles of centre 1, radii $\frac{\pi}{4}$ and $\frac{\pi}{3}$

Q6 (D), by completing the square $x^2 - 6x + 5 = (x-3)^2 - 4$

Q7 (C), $v = r\omega$, where $\omega = 2\pi f = 2\pi \times 10 = 20\pi$,

$\therefore v = 100\pi$

Q8 (C), Area $= (2y)^2 = 4y^2 = 4(16 - x^2)$

Q9 (B)

Q10 (A), $=$ total $-$ (6 pp in a room) $-$ (5 pp in a room)

$= 4^6 - {}^4C_1 - {}^4C_1 {}^6C_5 {}^3C_1 = 4020$

Question 11

(a) (i) $z + \overline{w} = 2 - i\sqrt{3} + (1 - i\sqrt{3}) = 3 - 2i\sqrt{3}$

(ii) $w = 2\,\text{cis}\frac{\pi}{3}$.

(iii) $w^{24} = 2^{24}\,\text{cis}\,8\pi = 2^{24}$

(b) $\frac{x^2+8x+11}{(x-3)(x^2+2)} = \frac{4}{x-3} + \frac{-3x-1}{x^2+2}$,

$\therefore A = 4, B = -3, C = -1$

(c) $z^2 + 4iz + 5 = (z-i)(z+5i)$

(d) $\int_0^1 x^3\sqrt{1-x^2}\,dx$

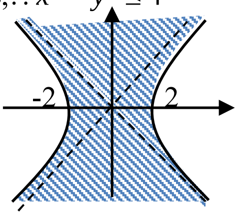
Let $u^2 = 1 - x^2, udu = -xdx.$

When $x = 0, u = 1$; when $x = 1, u = 0$

$I = -\int_1^0 (1-u^2)u^2\,du = \int_0^1 (u^2 - u^4)\,du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 = \frac{2}{15}$

(e) Let $z = x + iy, z^2 + \overline{z}^2 = (x+iy)^2 + (x-iy)^2$

$= 2(x^2 - y^2) \le 8, \therefore x^2 - y^2 \le 4$



Question 12

(a) Let $t = \tan\frac{x}{2}, dt = \frac{1}{2}\sec^2\frac{x}{2}dx, \therefore dx = \frac{2dt}{1+t^2}$

When $x = 0, t = 0$. When $x = \frac{\pi}{2}, t = 1$

$\int_0^{\frac{\pi}{2}} \frac{1}{4 + 5\cos x}dx = \int_0^1 \frac{1}{4 + \frac{5(1-t^2)}{1+t^2}} \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{2}{9-t^2}dt = 2\int_0^1 \frac{1}{(3-t)(3+t)}dt = \frac{1}{3}\int_0^1 \left(\frac{1}{3-t} + \frac{1}{3+t}\right)dt$

$= \frac{1}{3}\left[\ln\frac{3+t}{3-t}\right]_0^1 = \frac{1}{3}\ln 2$

(b) Differentiate $\ln y - \ln(1000 - y) = \frac{x}{50} - \ln 3$

$\frac{y'}{y} + \frac{y'}{1000-y} = \frac{1}{50}$

$y'\left(\frac{1000 - y + y}{y(1000-y)}\right) = \frac{1}{50}$

$y' = \frac{y}{50}\frac{1000-y}{1000} = \frac{y}{50}\left(1 - \frac{y}{1000}\right)$

(c) Using the shells method

$\partial V \approx 2\pi rh\partial x = 2\pi(4-x)e^x\partial x$

$V = 2\pi\int_1^3 (4-x)e^x\,dx$

Let $u = 4 - x, du = -dx$

Let $dv = e^x, v = e^x$

$\therefore \int_1^3 (4-x)e^x\,dx = \left[(4-x)e^x\right]_1^3 + \int_1^3 e^x\,dx$

$= \left[(4-x)e^x + e^x\right]_1^3 = \left[(5-x)e^x\right]_1^3 = 2e^3 - 4e.$

$\therefore V = 2\pi\left(2e^3 - 4e\right) = 4\pi\left(e^3 - 2e\right)$

(d) (i) $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-c}{p^2}}{c} = -\frac{1}{p^2}$

Eqn of tangent at $P, y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$\therefore x + p^2y = 2cp$

(ii) Let $x = 0, y = \frac{2c}{p}, \therefore B\left(0, \frac{2c}{p}\right)$

Let $y = 0, x = 2cp, \therefore A(2cp, 0)$

Midpoint of AB has coordinates $x = \dfrac{0+2cp}{p} = cp$,

$y = \dfrac{\frac{2c}{p}+0}{2} = \dfrac{c}{p}$, which are the coordinates of P.

$\therefore PA = PB$

Since $\angle AOB = 90°$, A, B and O are on a circle with centre P.

(iii) Similarly, $QD = QC$

$\therefore \dfrac{AP}{AB} = \dfrac{DQ}{DC} = \dfrac{1}{2}$

$\therefore AD \,//\, PQ \,//\, BC$ (When parallel lines are cut off by 2 transversals, the cut-off segments in one transversal are proportional to the corresponding segments in the other transversal)

Question 13

(a) (i) $I_n = \displaystyle\int_0^1 (1-x^2)^{\frac{n}{2}}\,dx$

Let $u = (1-x^2)^{\frac{n}{2}}, du = -nx(1-x^2)^{\frac{n}{2}-1}dx$

Let $dv = dx, v = x$

$$I_n = \left[x(1-x^2)^{\frac{n}{2}}\right]_0^1 + n\int_0^1 x^2(1-x^2)^{\frac{n}{2}-1}dx$$

$$= n\int_0^1 (x^2-1+1)(1-x^2)^{\frac{n}{2}-1}dx$$

$$= -n\int_0^1 (1-x^2)^{\frac{n}{2}}dx + n\int_0^1 (1-x^2)^{\frac{n}{2}-1}dx$$

$= -nI_n + nI_{n-2}$.

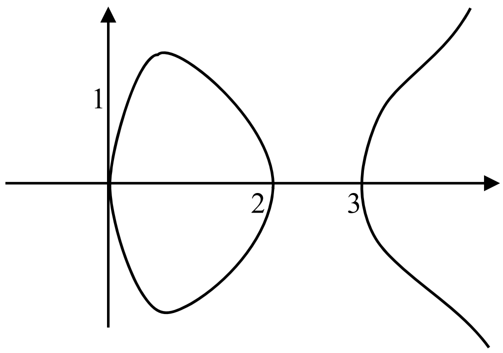
$\therefore (n+1)I_n = nI_{n-2}$.

$\therefore I_n = \dfrac{n}{n+1}I_{n-2}$.

(ii) $I_5 = \dfrac{5}{6}I_3, I_3 = \dfrac{3}{4}I_1, I_1 = \displaystyle\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}$ area of a circle of radius $1 = \dfrac{\pi}{4}$.

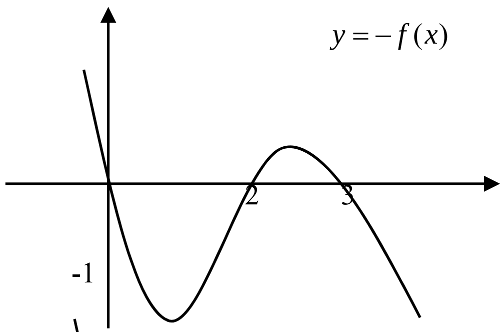
$\therefore\ I_5 = \dfrac{5}{6}\dfrac{3}{4}\dfrac{\pi}{4} = \dfrac{5\pi}{32}$

(b) (i) $y^2 = f(x)$

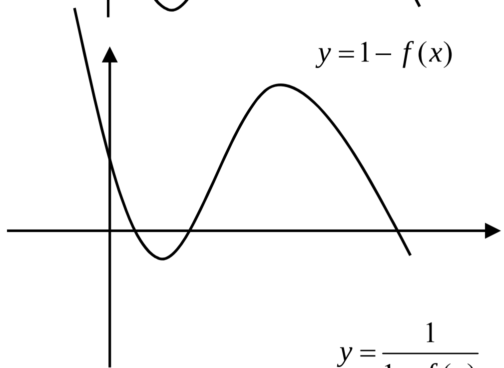


(ii) $y = \dfrac{1}{1-f(x)}$

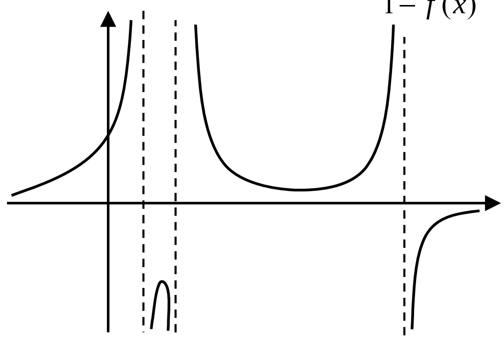
$y = -f(x)$



$y = 1 - f(x)$



$y = \dfrac{1}{1-f(x)}$



(c) (i) $\angle ABC = \angle ABD + \angle DBC = \alpha + \beta$

since $\angle ABD = \angle ACD = \beta$ and $\angle DBC = \angle DAC = \alpha$

(angles subtending the same arc are equal)

$\angle ACB = 90°$ (semi-circle angle)

$\therefore \sin(\alpha+\beta) = \dfrac{AC}{AB} = \dfrac{AC}{2r}$

$\therefore AC = 2r\sin(\alpha+\beta)$.

(ii) In ΔABD (noting $\angle ADB = 90°$, semicircle angle),

$\sin\beta = \dfrac{AD}{AB} = \dfrac{AD}{2r}, \therefore AD = 2r\sin\beta$

In $\Delta ADE, \cos\alpha = \dfrac{AE}{AD}$,

$\therefore AE = AD\cos\alpha = 2r\cos\alpha\sin\beta$

(iii) Similarly, $EC = 2r\cos\beta\sin\alpha$

In $\Delta ABC, \sin(\alpha+\beta) = \dfrac{AC}{AB} = \dfrac{AE+EC}{2r}$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Question 14

(a) Exact area $= \int_1^t \ln x dx = \left[x\ln x - x\right]_1^t = t\ln t - t + 1$.

Area of triangle $= \dfrac{1}{2}(t-1)\ln t$

Exact area > Area of triangle, $\therefore t\ln t - t + 1 > \dfrac{1}{2}(t-1)\ln t$

$2t\ln t - 2t + 2 > (t-1)\ln t$

$(t+1)\ln t > 2t - 2$

$\therefore \ln t > \dfrac{2(t-1)}{t+1}$ for $t > 1$

(b) Let $n = 2, |z_2| = |1+i| = \sqrt{2}, \therefore$ true for $n = 2$.

Assume $|z_n| = \sqrt{n}$ for some integer n.

Required to prove that $|z_{n+1}| = \sqrt{n+1}$.

$z_{n+1} = z_n\left(1 + \dfrac{i}{|z_n|}\right) = z_n\left(1 + \dfrac{i}{\sqrt{n}}\right)$

$\therefore |z_{n+1}| = \sqrt{n}\sqrt{1+\dfrac{1}{n}} = \sqrt{n}\sqrt{\dfrac{n+1}{n}} = \sqrt{n+1}$.

$\therefore$ By the principle of Mathematical Induction, it's true for all $n \geq 2$.

(c) (i) $\sec^{2n}\theta = (1+\tan^2\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta$

(ii) $\sec^8\theta = \sec^6\theta\sec^2\theta = \sum_{k=0}^{3}\binom{3}{k}\tan^{2k}\theta\sec^2\theta$

$= \left\{\binom{3}{0} + \binom{3}{1}\tan^2\theta + \binom{3}{2}\tan^4\theta + \binom{3}{3}\tan^6\theta\right\}\sec^2\theta$

$\therefore \int\sec^8\theta d\theta = \binom{3}{0}\tan\theta + \binom{3}{1}\dfrac{\tan^3\theta}{3} + \binom{3}{2}\dfrac{\tan^5\theta}{5}$

$+ \binom{3}{3}\dfrac{\tan^7\theta}{7} + C$

$= \tan\theta + \tan^3\theta + \dfrac{3}{5}\tan^5\theta + \dfrac{1}{7}\tan^7\theta + C$

(d) (i) $\angle A$ is common

$\dfrac{AD}{AC} = \dfrac{3}{6} = \dfrac{1}{2}, \dfrac{AE}{AB} = \dfrac{4}{8} = \dfrac{1}{2}, \therefore \dfrac{AD}{AC} = \dfrac{AE}{AB}$

$\therefore \Delta ABC ||| \Delta AED$ (SAS)

(ii) $\angle B = \angle AED$ (corresponding angles in similar Δs)

$\therefore BCED$ is a cyclic quad (interior angle = opposite exterior angle)

(iii) $\cos\angle DEA = \dfrac{3^2+4^2-3^2}{2\times3\times4} = \dfrac{2}{3}$

$\therefore \cos\angle DEC = \cos(\pi - \angle DEA) = -\cos\angle DEA = -\dfrac{2}{3}$

$\therefore CD^2 = 3^2 + 2^2 - 2\times3\times2\times -\dfrac{2}{3} = 21$

$\therefore CD = \sqrt{21}$

(iv) Let O be the centre,

$\angle DOC = 2(\pi - \angle DEC)$ (angle at the centre is twice angle on the circumference)

$\cos\angle DOC = \cos 2(\pi - \angle DEC)$

$= \cos 2\angle DEC$

$= 2\cos^2\angle DEC - 1$

$= 2\times\dfrac{4}{9} - 1 = -\dfrac{1}{9}.$

In $\Delta DOC, CD^2 = OC^2 + OD^2 - 2OC\times OD\times\cos\angle DOC$

$\therefore 21 = r^2 + r^2 + \dfrac{2}{9}r^2 = \dfrac{20}{9}r^2$

$\therefore r^2 = \dfrac{189}{20}.$

$\therefore r = \sqrt{\dfrac{189}{20}} = \dfrac{3\sqrt{105}}{10}.$

Question 15

(a) Area $A = \dfrac{1}{2}|z||w|\sin(\theta-\phi)$, assuming $\theta > \phi$.

$z\overline{w} - w\overline{z} = |z|\text{cis}\,\theta|w|\text{cis}(-\phi) - |w|\text{cis}\,\phi|z|\text{cis}(-\theta)$

$= |z||w|(\text{cis}\,\theta\,\text{cis}(-\phi) - \text{cis}\,\phi\,\text{cis}(-\theta))$

$= |z||w|(\text{cis}(\theta-\phi) - \text{cis}(\phi-\theta))$

$= |z||w|2i\sin(\theta-\phi)$, since $\cos(\theta-\phi) = \cos(\phi-\theta)$

and $\sin(\theta-\phi) = -\sin(\phi-\theta)$

$\therefore z\overline{w} - w\overline{z} = 4iA.$

(b) (i) $P(x) = ax^4 + bx^3 + cx^2 + e$

$P(1) = -3, \therefore a+b+c+e = -3$ (1)

$P(-1) = 0, \therefore a-b+c+e = 0$ (2)

$P'(x) = 4ax^3 + 3bx^2 + 2cx$

$P'(-1) = 0, \therefore -4a+3b-2c = 0$ (3)

(1) − (2) gives $2b = -3, \therefore b = -\dfrac{3}{2}$

Sub to (3), $-4a - \dfrac{9}{2} - 2c = 0$

$\therefore 4a + 2c = -\dfrac{9}{2}$

(ii) When $x = 1, m = 4a + 3b + 2c = 3b + (4a+2c)$

$= 3\times -\dfrac{3}{2} - \dfrac{9}{2} = -9$

(c) (i) $0.7^4 = 0.24$

(ii) Pr(at least 3 cars complete 4 days)

$= 1 - \text{Pr}(0,1,2 \text{ cars complete 4 days})$

$= 1 - 0.76^8 - {}^8C_1 0.76^7 0.24 - {}^8C_2 0.76^6 0.24^2$

(d) (i) Terminal velocity occurs when $\dot{v} = 0, \therefore mg = kv^2$

$\therefore v_T = \sqrt{\dfrac{mg}{k}}$

(ii) When the ball goes up, $mv\dfrac{dv}{dx} = -(mg + kv^2)$,

taking the direction upward positive

$$-\int_u^0 \frac{mvdv}{mg+kv^2} = \int_0^H dx$$

$$H = -\frac{m}{2k}\left[\ln(mg+kv^2)\right]_u^0 = \frac{m}{2k}\ln\frac{mg+ku^2}{mg}$$

$$= \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{\frac{mg}{k}}\right) = \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right)$$

(iii) When the ball goes down, $mv\dfrac{dv}{dx} = mg - kv^2$

$$\int_0^w \frac{mv\,dv}{mg-kv^2} = \int_0^H dx$$

$$H = -\frac{m}{2k}\left[\ln(mg-kv^2)\right]_0^w = -\frac{m}{2k}\ln\frac{mg-kw^2}{mg}$$

$$= -\frac{v_T^{\,2}}{2g}\ln\left(1-\frac{w^2}{v_T^{\,2}}\right) = \ln\left(\frac{v_T^{\,2}}{v_T^{\,2}-w^2}\right)$$

But $H = \dfrac{v_T^{\,2}}{2g}\ln\left(1+\dfrac{u^2}{v_T^{\,2}}\right) = \dfrac{v_T^{\,2}}{2g}\ln\left(\dfrac{v_T^{\,2}+u^2}{v_T^{\,2}}\right)$

$$\therefore \ln\left(\frac{v_T^{\,2}+u^2}{v_T^{\,2}}\right) = \ln\left(\frac{v_T^{\,2}}{v_T^{\,2}-w^2}\right)$$

$$\therefore \frac{v_T^{\,2}+u^2}{v_T^{\,2}} = \frac{v_T^{\,2}}{v_T^{\,2}-w^2}$$

$$\therefore -v_T^{\,2}w^2 + v_T^{\,2}u^2 - w^2u^2 = 0$$

$$\therefore v_T^{\,2}w^2 + w^2v^2 = v_T^{\,2}u^2$$

$$\therefore \frac{1}{u^2} + \frac{1}{v_T^{\,2}} = \frac{1}{w^2}$$

Question 16

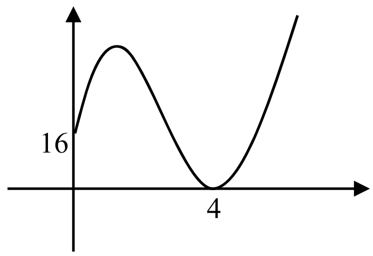
(a) $P(x) = 2x^3 - 15x^2 + 24x + 16$

$P'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4)$

$= 6(x-1)(x-4)$

$= 0$ when $x = 1, 4$

$P(0) = 16, P(1) = 27,$ and $P(4) = 0, \therefore$ For $x \ge 0$, the minimum of $P(x)$ is 0, which occurs at $x = 4$.



(ii) For $x \ge 0, 2x^3 - 15x^2 + 24x + 16 \ge 0$

$2x^3 + 10x^2 + 24x + 16 \ge 25x^2$

By trial and error, $2x^3 + 10x^2 + 24x + 16 = 0$ when $x = -1$.

$\therefore 2x^3 + 10x^2 + 24x + 16 = (x+1)(2x^2 + 8x + 16)$

$= (x+1)\left(x^2 + (x+4)^2\right).$

$\therefore (x+1)\left(x^2 + (x+4)^2\right) \ge 25x^2$

(iii) $\therefore x^2 + (x+4)^2 \ge \dfrac{25x^2}{x+1}$

Let $x = m+n, (m+n)^2 + (m+n+4)^2 \ge \dfrac{25(m+n)^2}{m+n+1}$

$\ge \dfrac{25(m^2+n^2+2mn)}{m+n+1} \ge \dfrac{25(2mn+2mn)}{m+n+1}$, since $m^2 + n^2 \ge 2mn$.

$\therefore (m+n)^2 + (m+n+4)^2 \ge \dfrac{100mn}{m+n+1}$

(b) (i) $PS + PS' = 2a, \therefore P$ lies on the ellipse with foci S and S'.

(ii) $SP = ePM$, where M lies on the directrix

$= e\left(\dfrac{a}{e} - a\cos\theta\right) = a(1 - e\cos\theta)$

(iii) $\sin\beta = \dfrac{QS'}{PS'} = \dfrac{a\cos\theta + ae}{a(1+e\cos\theta)} = \dfrac{e+\cos\theta}{1+e\cos\theta}$, as $SP = a(1-e\cos\theta)$ and $PS + PS' = 2a$.

(iv) Similarly, $\sin\angle QPS = \dfrac{QS}{PS} = \dfrac{e-\cos\theta}{1-e\cos\theta}$

Resolving the forces vertically,

$T\sin\beta - T\sin\angle QPS = mg$

$T\left(\dfrac{e+\cos\theta}{1+e\cos\theta} - \dfrac{e-\cos\theta}{1-e\cos\theta}\right) = mg$

$\therefore mg = \dfrac{T}{1-e^2\cos^2\theta}\Big(e+\cos\theta - e^2\cos\theta - e\cos^2\theta$

$-(e-\cos\theta + e^2\cos\theta - e\cos^2\theta)\Big)$

$= \dfrac{T}{1-e^2\cos^2\theta}\left(2\cos\theta - 2e^2\cos\theta\right)$

$= \dfrac{2T(1-e^2)\cos\theta}{1-e^2\cos^2\theta}$. (1)

(v) Resolving the forces horizontally,

$T\cos\beta + T\cos\angle QPS = mr\omega^2$

$T\left(\dfrac{PQ}{PS'} + \dfrac{PQ}{PS}\right) = mr\omega^2$

$\therefore mr\omega^2 = PQ \times T\dfrac{PS+PS'}{PS\times PS'}$

$= PQ\times T\dfrac{2a}{a(1-e\cos\theta)a(1+e\cos\theta)}$

$= PQ\times T\dfrac{2a}{a^2(1-e^2\cos\theta)}$

But $PQ = y_P = b\sin\theta = a\sqrt{1-e^2}\sin\theta$

$\therefore mr\omega^2 = \dfrac{2T\sqrt{1-e^2}\sin\theta}{1-e^2\cos\theta}$. (2)

(vi) $\dfrac{(2)}{(1)}$ gives $\dfrac{r\omega^2}{g} = \dfrac{\sqrt{1-e^2}\sin\theta}{(1-e^2)\cos\theta} = \dfrac{\tan\theta}{\sqrt{1-e^2}}$

$\therefore \tan\theta = \dfrac{r\omega^2\sqrt{1-e^2}}{g}$